Building Powerful Numeracy for Middle and High School Students

Pamela Weber Harris

HEINEMANN
Portsmouth, NH

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dedication

To my four children,
Abby, Craig, Matthew, and Cameron,
who have taught me so much.
You are my best guinea pigs!

contents

introduction

I can remember trying to reteach basic facts in general math. It was almost like putting a coin into a soda machine and having the machine reject it; the coin just falls through. Teaching computation over and over again is much the same; students just pass through, and none of the ideas stick.

GAIL BURRILL, NCTM PRESIDENT'S MESSAGE, 1997

Carl Sandburg once said, "Arithmetic is where numbers fly like pigeons in and out of your head. . . . It is numbers you squeeze from your head to your hand to your pencil to your paper till you get the answer." We can chuckle at Sandburg's insight and humor, and as math teachers we can all certainly agree that this is not how we want our students to feel. Instead, we want them to develop flexibility with numbers, to look to the numbers first before they calculate, and to choose an elegant, efficient strategy given those numbers. We want them to have a deep understanding of place value and properties of operations, and a repertoire of strategies for computation based on these understandings. This repertoire should include the standard algorithms with an understanding of the place value and properties involved, while at the same time recognizing that, depending on the numbers, they are often not the most beneficial strategies to use.

Before handheld calculators, emphasis was placed on paper-and-pencil arithmetic, but in today's world mental arithmetic strategies and the ability to judge the reasonableness of an answer have risen in importance. When the numbers are messy and the calculations are too difficult and cumbersome to do mentally, we usually reach for the calculator, not pencil and paper. Using algorithms, the same series of steps with all problems, is antithetical to calculating with number sense. Calculating with number sense means that one should look at the numbers first and then decide on a strategy that is fitting—and efficient.

As secondary teachers, we are often frustrated by the lack of number sense in our students. Students seem to either reach for a calculator or just shrug and say, "I don't know" when asked simple arithmetic questions. They seem ill-prepared to learn higher math because they have not memorized basic facts. Many students make careless errors with nonsensical results, yet do not recognize how far off their answers are. We are in the age of *algebra for all*, yet we have students who were obviously never in the *arithmetic for all* movement.

Recently, pockets of elementary students have begun to experience reform mathematics efforts. These students may use strategies you do not recognize. The Common Core State Standards (CCSS) consistently call for students to use "strategies and algorithms" and be able to "understand and explain why the procedures work." (Common Core State Standards Initiative 2011) As more states implement the CCSS, more students will arrive with alternative strategies. How do we support, not supplant, such learning?

This book is written for secondary mathematics teachers who wish students came to them with more arithmetic skills and more confidence in learning higher math. We want to encourage students to look not only at the operation in a problem but also at the numbers in the problem before they decide on a strategy. We also want them to have multiple strategies on hand to be able to check their solutions and have more confidence in their answers. We want students to have models in their heads with which to think about numbers and their relationships.

Since students don't always come to us that way, how can we help them? Solid research at the elementary level is showing us how to help all students be mathematically proficient by redefining what it means to compute with number sense, but since the literature is aimed at elementary teachers, it is largely unknown to secondary teachers. This book is an attempt to bridge that gap, to bring these insights to the secondary world.

For the last ten years, I have worked with this research and the resulting reform math materials with elementary teachers and students. As a graphing calculator expert, frequent secondary workshop presenter, and secondary curriculum writer and developer, I purposefully scrutinized it all, saw what worked and what didn't work, always with an eye to success in higher math. I was and continue to be amazed at the power we can harness in our secondary students by teaching ourselves and our students real numeracy.

The linchpin of this book is that when we help students construct numerical relationships, they begin to believe that mathematics is understandable, that it is not all about memorizing abstract, counterintuitive rules, but instead an arena in which they can reason and use their intuitive sense. We can develop their numeracy and use this understanding to build higher math. Chapter 1 provides a brief set of lessons learned from elementary reform. The rest of the chapters will focus on the main strategies, big ideas, and models for each operation; how those strategies, ideas, and models relate to higher mathematics; and how to help students construct these critical numerical relationships.

CHAPTER 1 *one* 1.

Numeracy

Anytime I am given a calculation problem (add, subtract, multiply, divide, percents, fractions . . . anything!), it's like I can see all these different ways to solve it running through my head like a ticker. Then I'm standing there trying to pick the quickest or most efficient way.

TEACHER AT NUMERACY WORKSHOP

Kim or Dana

Kim and Dana are two elementary teachers who have classrooms next door to each other. When I first began working with them, they told me very different stories about their math experience. Blushing, Kim told me that she did a lot of math in her head, but she assured me quickly that she knew and taught her students "the right way." Kim also stated, a bit shamefacedly, that, while she had gotten fine grades in math, her brother was really the mathy one. She could follow what her teachers wanted her to do, but she preferred to think about it her way. When Kim calculated, she played with the numbers until she found a strategy that made the problem as easy as she could.

Dana, on the other hand, followed what she called "all the rules." She made excellent grades in math, too, but by meticulously repeating exactly the steps that she was taught. Dana believed that mental math meant performing the algorithms in your head, and she claimed with conviction that people who were good at mental math were those who could "hold all of those numbers and the little slashes in their heads while they went through all of the right steps." She readily admitted that she wasn't good at (her version of) mental math, but that it didn't matter because she could always find a pencil if need be.

Do you see yourself in either of these reactions? Imagine that you are out at a rather nice restaurant for lunch with 11 friends. The check comes and the 12 of you decide to split the bill evenly. The bill is $249. You do a quick mental estimate of a 15% tip, add it on, and round up to $300. Now, how would you do the division to find out how much you each owe? Do you perform the standard long division algorithm on paper or in your head, as Dana would do, starting with "12 goes into 30"? Or do you play with the numbers, as Kim would do, perhaps seeing it immediately as equivalent to 100 divided by 4? When you were in school and the teacher said, "Show me your work," did you carefully mimic the teacher's procedures? Or were you more likely to play with the numbers?

I was a Dana, through and through. I made the grades and set the curves, but I had very little conceptual understanding of number. If I was in the grocery store and didn't have a pencil or calculator or couldn't hold all of those numbers in my head as I tried to figure the price per ounce, then I had a strategy. My strategy was simply to buy the bigger box. Surely the bigger box had to be the better buy, right? I, like Dana, thought that people good at mental math could do long division in their heads. I simply had no concept that there was a whole realm of numerical relationships that I could play with, manipulate, and use to my advantage to solve problems.

If you are more like Kim, this book offers you a structure and framework to guide students who are more like Dana and to help students who are like yourself to expand their natural tendencies. If you are like Dana, you can sympathize with your students who may think, as you once did, that mental math means doing algorithms in their heads and help these students develop the numerical power that will allow them to move away from these "imaginary" algorithms, while helping the Kims in your class build their numerical power. Helping students learn to choose appropriate strategies and models, and developing their understanding of why the procedures work, will build your students' numeracy.

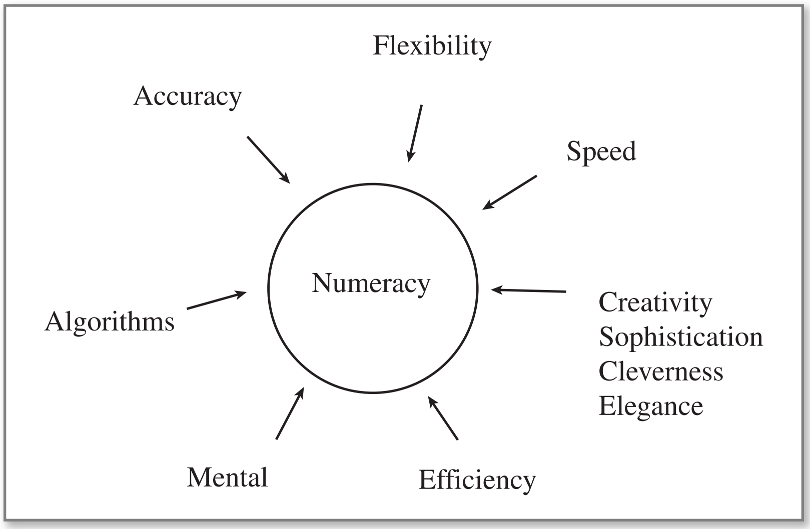
Understanding Numeracy

All humans are born with the perceptual intuitive ability to compare amounts and to even see small amounts like two or three as subunits, or groups, without needing to count them (Dehaene 1999). As children learn to count, they construct the idea that the counting numbers grow by 1 (though this linear relationship usually breaks down for them as the numbers grow larger). Unfortunately, children's early intuitive number sense was not discovered until recently, and so rather than building upon it, well-meaning teachers have often worked against it by emphasizing procedures and sacrificing the development of number relations. As a result, many learners (like Dana) abandoned their intuitive math sense in order to adopt the procedures being emphasized. They lose sight of how the numbers are related to each other.

Researchers now believe that people with good number sense (like Kim) have capitalized on their own natural innate math sense despite the emphasis placed on procedures in most schools (Stein 2008). Over the last several years, pockets of elementary students have begun to experience mathematics based more on numeracy and less on algorithms. You may have some of these students, but others may still come to you without this strong foundation.

Numeracy, at its heart, is using mathematical relationships to reason with numbers and numerical concepts. The diagram in Figure 1.1 shows the components of numeracy.

FIGURE 1.1



Flexibility

We want students who will look at the numbers in a problem and consider more than one way to solve the problem. This flexibility allows students to have more confidence in their solution because they see relationships in the numbers more clearly.

Efficiency

Give students experience with different strategies and problem types so that they will know which kinds of problems lend themselves to which strategies. We want students to consider the numbers in a problem, not just the operation, before they decide on an efficient strategy to solve it.

Accuracy

Getting the right answer has traditionally been the focus of math instruction. While accuracy is important, it is not the only part of numeracy that matters. As students begin exploring which strategies work nicely for certain numbers with an eye for clever, efficient, or even elegant strategies, they will actually get more answers correct because the emphasis is on relationships.

Creativity, Sophistication, Cleverness, and Elegance

"Mathematics is a highly creative activity. Mathematicians solve problems but they also pose problems. They inquire. They explore relations, investigate interesting patterns, and craft proofs. They present their ideas to the mathematics community and those ideas hold up only when the logic of the argument is accepted" (Fosnot et al. 2008; www.contextsforlearning.com/seriesOverview.asp). When mathematicians write proofs, they seek to make their work clean, concise, and understandable. The process is much less about brute force and much more about creativity and elegance. When students compute, they can seek clever and sophisticated strategies based on the relationships they are learning. They can play with the numbers, bringing their creativity to bear.

Algorithms

The common arithmetic algorithms have a place in numeracy. We can study why they work, how place value is used to create such compact processes that work for all such problems. Indeed, that each algorithm works for the entire class of problems is a wonderful historic feat. The invention of these algorithms allowed even those not trained in the use of the abacus to compute. We should study why these processes work. However, how often should we actually use them? In today's world, we have other, even quicker and far more accurate methods of computing (such as calculators and spreadsheets) when we are faced with dealing with many large numbers. If the algorithms are students' only computation strategy, the algorithms' *digit approach* may inadvertently affect students' progress in higher math. It is possible for students to be successful with the addition and subtraction algorithms using counting strategies instead of additive strategies and with the multiplication and division algorithms using additive instead of multiplicative reasoning. If students have not developed additive and multiplicative reasoning, it will be very difficult for them to reason proportionally and algebraically as they will need to in secondary mathematics.

In a study of forty-four mathematicians' computation strategies, Ann Dowker (1992) found that "mathematicians tended to use strategies involving the understanding of arithmetical properties and relationships rather than strategies involving the use of school-taught techniques" (52). To these mathematicians, solving these problems "seemed to involve an enjoyment of thinking about and playing with numbers, rather than rote memorization" (52). For all the mathematicians and all the problems, they only used an algorithm 4% of the time. "Most mathematicians used a wide variety of strategy types during the task and, if retested, often used different strategy types for the same problem on the two occasions" (53).

Speed

We, as secondary math teachers, were generally very good at the algorithms. We practiced them and have used them successfully for a long time, so naturally, they will be faster for us initially. But with a little practice, more often than not, a more transparent strategy is at least as fast as the algorithm. The sophisticated strategies in this book combine both speed and greater understanding of the mathematics involved.

Mental

Doing mental arithmetic does not mean doing it all in your head. Instead, it implies that you are using your head to reason. Many who employ the strategies suggested in this book use paper and pencil to keep track of their mental steps. This is different from writing down numbers and performing a predetermined set of steps that have been memorized without understanding.

The Importance of Representation

How do we build numeracy? One key lesson from elementary reform efforts is the importance of representation. The representation of student strategies on models such as the open number line, the open array, and the ratio table promotes discussion on relationships rather than on procedures. For example, consider the subtraction problem 36 − 19. Before you read on, find an answer. What strategy did you use?

Several strategies for solving this problem can be shown on an open number line. Perhaps you subtracted an "easy" number first: 36 − 16 − 3 = 17; or 36 − 10 − 9 = 17. This strategy can be modeled on an open number line as shown in Figure 1.2. You could over-subtract and adjust: 36 − 20 + 1 = 17. On an open number line this would be modeled as in Figure 1.3.

FIGURE 1.2

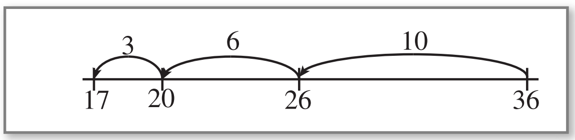
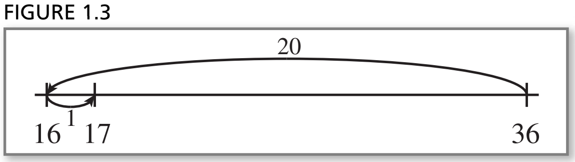
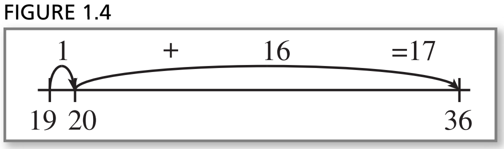


FIGURE 1.3



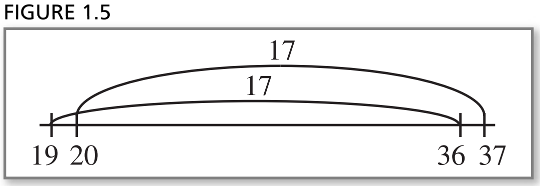
Many people would rather add up than remove. They know that 36 − 19 can be thought of as "19 plus something is 36." So 19 + 1 = 20, then 20 + 16 = 36. This can also be modeled on the number line as in Figure 1.4.

FIGURE 1.4



Some people would rather keep the difference constant but make the problem easier, turning 36 − 19 into 37− 20. This can be modeled on the number line as shown in Figure 1.5.

FIGURE 1.5

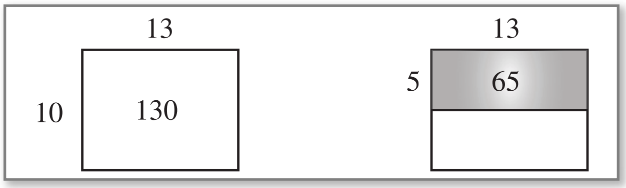


When you model student strategies, it is a model *of thinking*. As students *see their thinking*, they can begin to use the model as tool to solve problems, a tool *for thinking*.

(Gravemeijer 1999). Students begin to operate using the model. Then students continue to develop more sophisticated strategies using the model as a tool, as we'll explore with two additional examples.

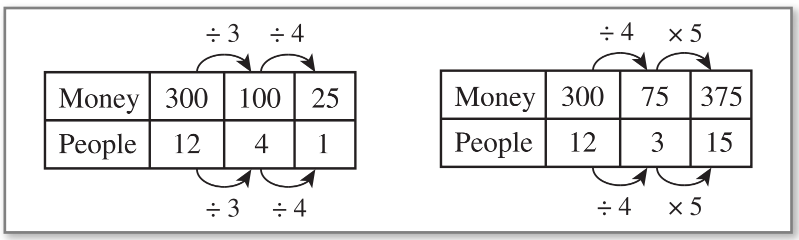
For multiplication, arrays are useful models. For example, consider 5×13. One strategy for solving this problem is to start with a 10×13 array. For many people, finding 10×13 is easy: 130. If you cut that array in half, you can see that 5×13 is half of 10×13, or 65. The array in Figure 1.6 models this strategy.

FIGURE 1.6



A ratio table is a useful tool for division. For example, remember how you and 11 friends split a \$300 tab? Consider the ratio of \$300/12 people. This can be simplified to \$100/4 people and \$25/person. Note that with the ratio table (see Figure 1.7), we could also answer a proportion question such as, "If a meal costs \$300 for 12 people, how much would it cost for 15 people?"

FIGURE 1.7



Using a number line, array, and ratio table requires an understanding of number relations, getting right to the heart of numeracy.

Problem Strings

In my experience, many well-meaning teachers ask students to show their work. But they do not really mean, "Show *your* work." They actually mean, "Show *my* work. Mimic *me*. Show all of *my* steps in solving that problem." These teachers may have found success in school mathematics by carefully memorizing and painstakingly imitating the rules and procedures they were taught. Like me, they had been successful with just such learning.

But is the best way to *do* a problem necessarily the best way to *learn about* solving those kinds of problems, to learn about the concepts involved? What does it mean for us to skip over the *understanding* part to the *doing* part? What do we lose or gain in the process? Superimposing a rule, even a "good" rule, can force learners to abandon the meaning they were starting to construct. Thus we end up with some students who can solve textbook problems very nicely, but not problem solvers who can adapt their understanding to new contexts and situations. We also end up with other students who believe they can't solve problems because for them, the memorized rules never worked.

By allowing students to solve problems in their own way and then modeling, comparing, and discussing different strategies with the rest of the class, we honor the students' thinking and nudge students toward more efficient and sophisticated thinking. One way of achieving this result is through problem strings.

A *problem string* is a purposefully designed sequence of related problems that helps students mentally construct numerical relationships and nudges them toward a major, efficient strategy for computation. This method of teaching was introduced by Cathy Fosnot and her colleagues (2008) in the Young Mathematicians at Work series. "Our goal in designing these strings was to encourage children to look to the numbers first before they decided on a strategy and to develop a sense of landmark numbers and a toolbox of strategies in order to calculate efficiently and elegantly—like mathematicians who employ a deep understanding of number and operation." Fosnot and her colleagues have written at length on strings for elementary and middle school students in Young Mathematicians at Work and the Contexts for Learning series. As I have used their materials extensively with students and teachers for the past several years, I have adapted some of their strings and added many of my own.

In general, the following approach works well for introducing problem strings. Depending on how easy the first few problems are for your students, give more or less wait time.

- The first problem is often an easy one, the foundation to build on. Write it on the board and ask students for the answer and how they solved the problem. Write the answer and model the strategy.
- Write the next problem and give appropriate think time. Encourage students who succeed with one strategy quickly to try to find a more efficient or sophisticated strategy to solve the same problem. Note that it is always important for students to find the answer themselves first, before other students begin explaining their strategies. Each student must have a sense of the relationships between the numbers in the problem before seeing other strategies. Also note that we want students to show their strategies so we can understand their thinking.
- Walk around and find a student who has used the goal strategy and one or two students who used different strategies.
- Ask the one or two other students to explain as you model their strategies on the board. Then ask the student who used the target strategy to explain as you model the strategy. For each strategy, ask students how many of them used that strategy.
- Continue the same approach with the other problems in the string.
- As you reach the end of the string, you may want to ask for fewer strategies to be shared and instead focus on the target strategy, depending on the length of the string and, more importantly, where students are in their understanding. If this is their first time with the strategy, then you might take longer and elicit more strategies. If students have worked with this strategy before, you might spend less time on extraneous strategies and focus on what makes a strategy suitable for a given problem.
- We want to encourage students to be guided by the numbers in the problem to find efficient and even clever or elegant solution strategies. The goal is for students to

make connections between the numbers in the problem and different strategies; if you just start telling students which strategies go with which kinds of problems, it will become just another bunch of tricks to memorize. We want students to construct the relationships and be able to use them fluently, not memorize them. Our goal is to learn as much mathematics as we can, with as little memorization as possible.

- The last problem is usually a bit different than the other problems, using bigger or less friendly numbers, where the connection to the target strategy might not be as obvious. By this point, some students will see the connections and use the target strategy. Do not fret about the other students. With more time and experience, they will continue building their numerical power. The goal is not simply to have students discover the strategy, but for them to construct numerical relationships and to begin to use those relationships to solve problems.

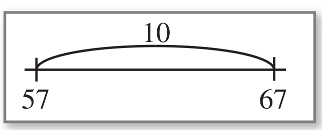
Figure 1.8 shows an example of a string designed to teach a particular addition strategy, that of going slightly over and adjusting to find the correct sum. The following steps show how you might use this string in class.

FIGURE 1.8

An Example: Addition Over Strategy
57 + 10
57 + 9
57 + 19
57 + 49
46 + 39
46 + 99
215 + 495

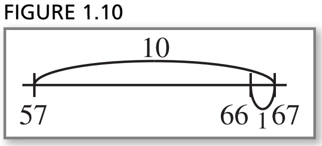
1. Write *57 + 10*, pause briefly, ask students for the answer, and write = *67*. This problem is easy, but to set up the pattern for the rest of the string, model the problem on an open number line, as shown in Figure 1.9.

FIGURE 1.9



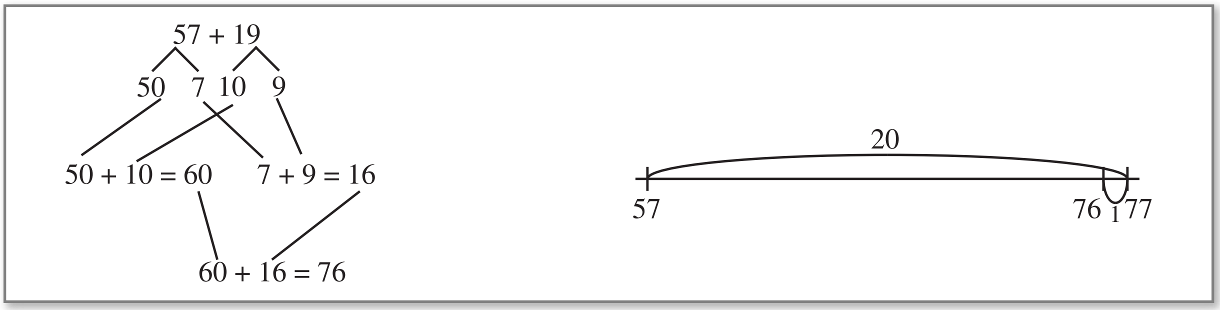
2. Write *57 + 9* on the board, and ask students for the answer. Ask if anyone used the previous problem to help. Again, this is easy, but draw the model (Figure 1.10) and then write the answer, = *66*.

FIGURE 1.10



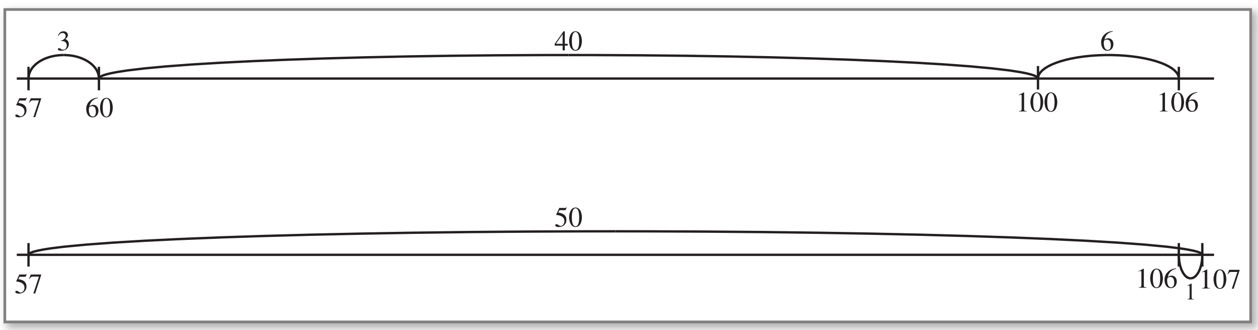
3. Give students the next problem, 57 + 19. Circulate and find someone who used the "over" strategy and someone who used another strategy, such as splitting the numbers into place-value chunks. Encourage students to show their thinking.
4. Bring the group back together for discussion. Ask the student who used the splitting strategy to describe what he or she did, write the answer, and model the strategy. Then do the same for the student who used the target "over" strategy. See Figure 1.11 for the models.

FIGURE 1.11



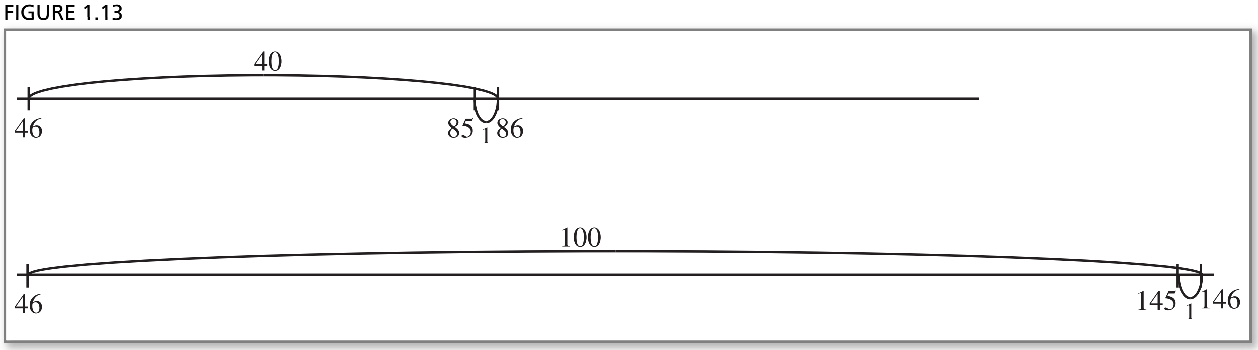
5. Repeat with the next problem, 57 + 49. As before, find a student who used the target strategy and one who used a different strategy. Model both strategies (as in Figure 1.12) as students discuss them.

FIGURE 1.12



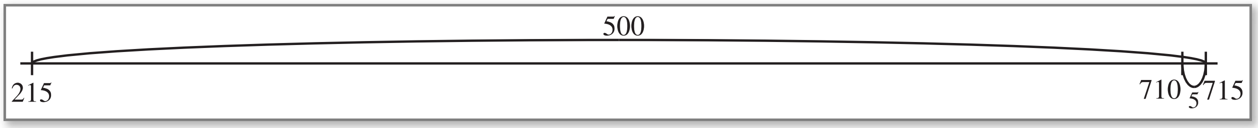
6. Continue with the next two problems, focusing on the over strategy. Model students' work as in Figure 1.13.

FIGURE 1.13



7. Encourage students to discuss what they have been doing. Have them turn and talk to a partner about the strategy they've seen. Then bring students back together and discuss as a group.
8. After students have discussed the strategy, give them the final, more challenging problem, 215 + 495. When you've given them time to think, have them discuss the strategy as you model it (Figure 1.14). Then ask students what they might call this strategy, developing language they can use in later discussions.

FIGURE 1.14



This string encourages using the relationships of the numbers to find sums. Note that while the teacher mostly modeled the target over strategy, students were asked to solve the problems in ways that made sense to them. The discussion promotes numeracy as students see strategies modeled, compared, and discussed. The teacher supports the students' understanding by generalizing why the over strategy works well for the numbers in these particular problems.

If some students have not tried your target strategy by the end of a string, encourage them to do so, but don't force anyone. They may not be ready now, but they may be the next time around. Problem strings for a particular strategy are meant to be done more than once, and each time students will bring to the strings more and more numerical power. As Fosnot and Uittenbogaard (2007, 10) caution, "The intent is not to get all learners to use the same strategy at the end of the string. That would simply be discovery learning. The strings are crafted to support development of computational fluency, to encourage students to look at the numbers and to use a variety of strategies helpful for working with those numbers." Note that the problems and problem strings are not intended to be used all at once, handed out as worksheets, or used as independent work for students.

In the chapters that follow, you will find examples of problem strings for different operations useful to secondary students, as well as some guidelines for creating your own problem strings.

By using models and problem strings in a systematic way, with an eye toward the major, efficient strategies, we can help secondary students construct mental numerical relationships. For those students who come to us with some elementary work in numeracy, we can build on their current understanding and support their continued development, while at the same time helping those who lack that understanding gain the foundation they need. When we give secondary students this numerical power, we also help them engage in learning higher mathematics with more confidence and more success.